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A note on the sensitivity of the strategic asset allocation problem

W.J. Hurley *, Jack Brimberg

Department of Mathematics and Computer Science, Royal Military College of Canada, Canada

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ABSTRACT

The Markowitz mean–variance portfolio optimization problem is a quadratic programming problem whose first-order conditions require the solution of a linear system. It is well known that the optimal portfolio weights are sensitive to parameter estimates, particularly the mean return vector. This has generally been attributed to the interaction of estimation error and optimization. In this paper we present some examples that suggest the linear system produced by the first-order conditions is ill-conditioned and it is this property that gives rise to the sensitivity of the optimal weights.

1. Introduction

The mean–variance portfolio optimization problem dates to the pioneering work of Markowitz [1]. It is well known that the optimal portfolio weights are sensitive to the parameter input, particularly the mean return vector. See, for the example, the work of Best and Grauer [2], Broadie [3], Chopra [4], Chopra and Ziemba [5], Frankfurter et al. [6], and Michaud [7]. This sensitivity has generally been attributed to the tendency for optimization to magnify the effects of estimation error. For this reason, Michaud [7] has referred to "portfolio optimization" as "error maximization".

There is now a vast literature on how to deal with the problem. As might be expected, the literature focuses on improved estimation procedures and model variations. Efforts to improve parameter estimation procedures include the work of Jobson and Korkie [8] and Jorion [9,10] on shrinkage estimators and Ledoit and Wolf [11] on reducing the error in the estimation of the covariance matrix. A host of researchers look at robust portfolio optimization (see, for example, Goldfarb and Iyengar [12], Garlappi et al. [13], and Lu [14]). Different formulations of the problem also include the work of Black and Litterman [15], Konno and Yamazaki [16], Simaan [17], and more recently, Jangannathan and Ma [18] and DeMiguel et al. [19]. It is important to clarify what financial theorists mean when they refer to robust portfolio optimization. In the general sense, robust optimization implies finding solutions that can be modified later in an effective manner once actual conditions are known. However, with portfolio management, the input parameters are consistently changing, and robustness in this setting refers to finding solutions that are insensitive to these changes. That is, portfolio formation strategies are sought that are relatively immune to variations in input values. Here, we offer no solution to the problem. Rather, we make a simple but important point. It is not always true that optimization magnifies estimation error and we show this using the basic Economic Order Quantity Model. The implication is that there has to be a deeper explanation of why portfolio weights are so sensitive to estimation error. In our view, this explanation has to do with *the underlying structure of the model*. We argue that the first-order conditions of the Markowitz portfolio optimization model result in a linear system that is ill-conditioned and *it is this poor conditioning that leads to the extreme sensitivity of the portfolio weights*.

This observation has a number of important implications. First, it is very unlikely that improved estimation techniques will solve the problem. DeMiguel et al. [20] took an exhaustive look at how existing improved estimation procedures and different models stacked up against a naive $1/n$ portfolio (a portfolio with funds divided equally among n assets). They found that none of these offered any significant performance improvement based on standard measures (including the Sharpe ratio and certainty-equivalent return). This is consistent with our observation on conditioning that we are not likely to find a magic bullet to solve the problem.

The work of Ledoit and Wolf [11] is particularly interesting. They consider only the covariance matrix. They offer the equivalent of a Stein estimator (i.e. a shrinkage estimator) for the covariance matrix and show that it has nice consistency properties as the dimension of the problem (both in the dimension of the covariance matrix and the dimension of the data set used to estimate it) gets large. Their estimator performs reasonably well and the condition of the covariance matrix falls dramatically for problems with a large number of assets and a large dataset. However, there are problems of interest where there is no guarantee that the dimension of the covariance matrix will be high. We have in mind the

* Corresponding author. Tel.: +1 613 541 6000x6468, +1 613 449 6120 (mob).
E-mail address: hurley_w@rmc.ca (W.J. Hurley).

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Strategic Asset Allocation Problem (SAAP) where only a handful of global asset classes is considered. An example is the problem considered by Black and Litterman [15].

2. It is not just optimization magnifying estimation error

Among others, Michaud [7] has argued that the sensitivity of the portfolio optimization problem is due to optimization magnifying estimation error. But this cannot be the complete explanation. After all there are many examples of optimization problems where slight errors in the parameter input are not magnified by optimization. For instance consider the Economic Order Quantity model used for inventory decisions. This model argues that the order size that minimizes transaction costs is

$$x^* = \sqrt{kD} \tag{1}$$

where x^* is the order size, k is a parameter that depends on the costs of holding and processing inventory, and D is the demand rate for the inventory over the period under consideration. Now suppose a small error ε is made when D is estimated. Then we have that

$$x^*(\varepsilon) = \sqrt{kD(1+\varepsilon)} \tag{2}$$

and the relative error is

$$\frac{x^*(\varepsilon) - x^*}{x^*} = \frac{\sqrt{kD(1+\varepsilon)} - \sqrt{kD}}{\sqrt{kD}} \simeq \varepsilon/2. \tag{3}$$

Thus, a 10% error in the estimation of demand, leads to approximately a 5% change in the recommended inventory level. The implication is that an explanation of solution sensitivity *must appeal to the underlying structure of the problem.*

3. The sensitivity of the SAA portfolio

Here is an example which demonstrates the sensitivity of the SAAP. Suppose an investor is considering a portfolio of three funds: LargeCap Equity; Foreign; and Bond. The investor estimates that expected returns for these are:

Asset	**Return**
LargeCap	0.1213
Foreign	0.1548
Bond	0.0923

(4)

and the covariance matrix is

	LargeCap	*Foreign*	*Bond*
LargeCap	0.02528	0.02098	0.00411
Foreign	0.02098	0.05452	0.00085
Bond	0.00411	0.00085	0.00487.

(5)

Let the random return of asset i, r_i, be normally distributed with mean $\bar{r}_i$ and variance σ_i^2. Let the covariance of the returns on assets i and j be c_{ij}. Suppose the investor is considering a portfolio where a proportion, x_i, of his total investment will go into asset i. The expected return on the portfolio is

$$\bar{r}_p = \bar{r}_1 x_1 + \bar{r}_2 x_2 + \bar{r}_3 x_3 \tag{6}$$

and, for convenience, we define the risk measure to be 1/2 the variance of the portfolio return:

$$\Phi(x_1, x_2, x_3) = \frac{1}{2} Var(r_p) \tag{7}$$

$$= \frac{1}{2} x^T \Omega x \tag{8}$$

$$= \frac{1}{2}[\sigma_1^2 x_1^2 + \sigma_2^2 x_2^2 + \sigma_3^2 x_3^2 + 2c_{12}x_1x_2 + 2c_{13}x_1x_3 + 2c_{23}x_2x_3] \tag{9}$$

where $r_p = r_1x_1 + r_2x_2 + r_3x_3$ is the uncertain portfolio return, Ω is the covariance matrix, and $x = (x_1, x_2, x_3)^T$ is a vector of portfolio weights. This investor wishes to minimize the variance of his portfolio return subject to it producing a mean return r_0. Hence he will solve the following SAAP:

$$\begin{aligned} \min \quad & \Phi(x_1, x_2, x_3) \\ \text{s.t.} \quad & \bar{r}_1 x_1 + \bar{r}_2 x_2 + \bar{r}_3 x_3 = r_0 \\ & x_1 + x_2 + x_3 = 1. \end{aligned} \tag{10}$$

The first-order necessary conditions require a solution of a linear system:

$$\begin{bmatrix} \sigma_1^2 & c_{12} & c_{13} & -\bar{r}_1 & -1 \\ c_{12} & \sigma_2^2 & c_{23} & -\bar{r}_2 & -1 \\ c_{13} & c_{23} & \sigma_3^2 & -\bar{r}_3 & -1 \\ \bar{r}_1 & \bar{r}_2 & \bar{r}_3 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \lambda_r \\ \lambda_x \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ r_0 \\ 1 \end{pmatrix} \tag{11}$$

where λ_r and λ_x are Lagrange multipliers. The matrix of this system is called the *augmented covariance matrix* and we represent it with Ω_+.

The solution of the system with $r_0 = 0.135$ and the parameter input described in (4) and (5) is:

$$x_1 = 0.195, \qquad x_2 = 0.593, \qquad x_3 = 0.212. \tag{12}$$

Suppose now the expected return on the LargeCap asset class is changed from 12.13% to 13.34%, a change of 10%. Then the new portfolio weights are

$$x_1 = 0.503, \qquad x_2 = 0.352, \qquad x_3 = 0.145. \tag{13}$$

Note that the LargeCap weight, x_1, increases by 160%; the other two change by an average of 36%. So a small change in a single input parameter can give rise to substantial changes in the optimal portfolio weights.

4. The origin of the sensitivity

Consider the linear system

$$Ax = b \tag{14}$$

where A is an $n \times n$ matrix, and x and b are $n \times 1$ vectors. We assume that A is nonsingular. One definition of the *norm* of the matrix A is

$$\|A\| = \max_{x \neq 0} \frac{\|Ax\|}{\|x\|} \tag{15}$$

where

$$\|x\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2} \tag{16}$$

is the usual vector norm. It is easy to show that the definition (15) is equivalent to

$$\|A\| = \max_{\|x\|=1} \|Ax\|. \tag{17}$$

The *condition* of A, $\kappa(A)$, is defined as

$$\kappa(A) = \|A\| \times \left\|A^{-1}\right\|. \tag{18}$$

Suppose the matrix is perturbed from A to $A + \delta A$. This leads to the perturbed solution $x + \delta x$ so that

$$(A + \delta A)(x + \delta x) = b. \tag{19}$$

Hence we have two systems

$$(A + \delta A)(x + \delta x) = b \tag{20}$$

$$Ax = b. \tag{21}$$

Expanding the first gives

$$Ax + A\delta x + \delta Ax + \delta A\delta x = b \tag{22}$$

and substituting the second results in

$$A\delta x + \delta Ax + \delta A\delta x = 0, \tag{23}$$

which may be rewritten as

$$\delta x = -A^{-1}(\delta Ax + \delta A\delta x) = -A^{-1}\delta A(x + \delta x). \tag{24}$$

We can now take the norm of both sides:

$$\begin{aligned} \|\delta x\| &= \left\|-A^{-1}\delta A(x+\delta x)\right\| \\ &= |-1|\left\|A^{-1}\delta A(x+\delta x)\right\| \\ &= \left\|A^{-1}\delta A(x+\delta x)\right\|. \end{aligned} \tag{25}$$

For any square matrices B_1 and B_2, and any vector $y \neq 0$, we have

$$\|B_1 y\| = \frac{\|B_1 y\|}{\|y\|}\cdot\|y\| \leq \|B_1\|\,\|y\| \tag{26}$$

assuming the product $B_1 y$ is defined. Analogously, we get

$$\|B_1B_2y\| \leq \|B_1\|\,\|B_2y\| \leq \|B_1\|\,\|B_2\|\,\|y\| \tag{27}$$

again assuming the product B_1B_2y is defined. Applying (27) to the right-hand side of (25) gives

$$\begin{aligned} \left\|A^{-1}\delta A(x+\delta x)\right\| &\leq \left\|A^{-1}\delta A\right\|\,\|x+\delta x\| \\ &\leq \left\|A^{-1}\right\|\,\|\delta A\|\,\|x+\delta x\| \end{aligned} \tag{28}$$

and therefore

$$\|\delta x\| \leq \left\|A^{-1}\right\|\,\|\delta A\|\,\|x+\delta x\|. \tag{29}$$

Dividing both sides by $\|x+\delta x\|$ gives

$$\frac{\|\delta x\|}{\|x+\delta x\|} \leq \left\|A^{-1}\right\|\,\|\delta A\|. \tag{30}$$

Finally, we can multiply the right-hand side by $\|A\| / \|A\|$ to get

$$\frac{\|\delta x\|}{\|x+\delta x\|} \leq \kappa(A)\frac{\|\delta A\|}{\|A\|}. \tag{31}$$

That is, the relative error in the solution vector is bounded above by the condition of A multiplied by the relative error in the matrix A. Moreover, it can be shown that this inequality is tight (see [21]). That is, there exists a δA such that (31) holds with equality.

Now consider calculating the condition of the system in (11) using the input from (4) and (5). To get the matrix norms, we could solve two separate optimizations, but it just as easy to set up the optimization

$$\begin{aligned} \max \quad & \|\Omega_+ u\| + \left\|\Omega_+^{-1} v\right\| \\ \text{s.t.} \quad & \|u\| = 1 \\ & \|v\| = 1. \end{aligned} \tag{32}$$

Note that this program is separable in u and v. Denoting the optimal solution u^* and v^*, we have that

$$\|\Omega_+\| = \left\|\Omega_+ u^*\right\| = 1.768, \tag{33}$$

$$\left\|\Omega_+^{-1}\right\| = \left\|\Omega_+^{-1} v^*\right\| = 95.182, \tag{34}$$

and, therefore, the condition of Ω_+ is

$$\kappa(\Omega_+) = \|\Omega_+\|\left\|\Omega_+^{-1}\right\| = 168.3. \tag{35}$$

Suppose there is a 1% relative error in Ω_+, that is, $\|\delta\Omega_+\| / \|\Omega_+\| = 0.01$. Then the relative error in the solution can be as high as $\|\delta x\| / \|x + \delta x\| = 168.3 \times 0.01 = 1.683$, or over 150%! And this is true even before we start to talk about the nature of the estimation error. Thus the portfolio problem can be poorly conditioned,

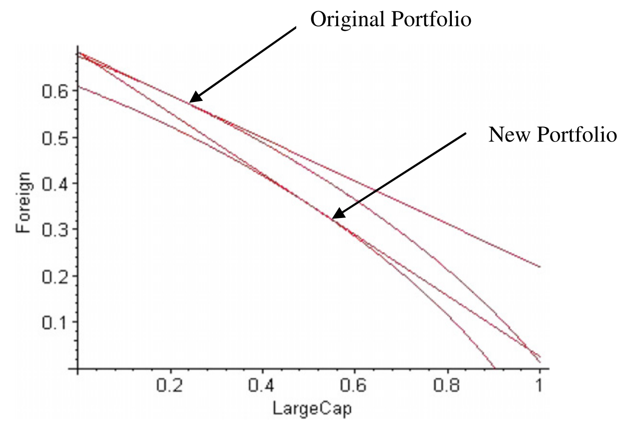

Fig. 1. The effects of changing the largecap return from 12.13% (original portfolio) to 13.34% (new portfolio).

and at least for some instances of the problem, small parameter estimation errors will lead to large changes in the optimal portfolio weights.

One way to interpret this ill-conditioning is to examine the optimization problem geometrically. The solution (portfolio weights) occurs where the hyperspace generated by the linear equality constraints is tangent to a contour of portfolio risk, which is an ellipse. It turns out that level sets of this quadratic function are extremely flat at these tangent points. What this means is that slight changes in the slope of the hyperspace or level sets of the objective function (i.e. slight changes in the expected return vector or the covariance matrix) can lead to significant changes in portfolio composition.

For the example above, a picture is shown in Fig. 1. The expected return constraint is shown in x_1x_2 space where it is tangent to the level set of portfolio risk at the optimum (this point is labelled "Original Portfolio"). Note that the level set (the curve) is very flat. When the LargeCap expected return is changed by 10%, the slope of the expected return constraint becomes steeper and is tangent to a new level set of portfolio risk (labelled "New Portfolio"). The resulting changes in portfolio weights are significant.

The condition of the minimum variance portfolio problem

A number of recent approaches (see [18,19]) have argued that the minimum variance portfolio has some nice performance properties. Unfortunately finding a minimum variance portfolio in the context of the SAAP is also ill-conditioned.

The minimum variance portfolio is found by solving

$$\begin{aligned} \min \quad & \Phi(x_1, x_2, x_3) \\ \text{s.t.} \quad & \sum_i x_i = 1. \end{aligned} \tag{36}$$

Clearly the linear system to be solved is again an augmented covariance matrix. Let this augmented matrix be denoted Ω_V.

To demonstrate the ill-conditioning of this problem, consider the following example taken from [22]. There are six assets: American Airlines (AMR), Bethlehem Steel (BS), General Electric (GE), International Harvester (HR), Philip Morris (MO), and Union Carbide (UK). The covariance matrix is

	AMR	BS	GE	HR	MO	UK
AMR	0.2060	0.0375	0.1077	0.0493	0.0208	0.0059
BS	0.0375	0.0790	0.0355	0.1028	0.0089	0.0406
GE	0.1077	0.0355	0.0867	0.0443	0.0194	0.0148
HR	0.0493	0.1028	0.0443	0.4435	0.0193	0.0274
MO	0.0208	0.0089	0.0194	0.0193	0.0083	−0.0015
UK	0.0059	0.0406	0.0148	0.0274	−0.0015	0.0392.

The expected returns are AMR(0.2032), BS(0.0531), GE(0.1501), HR(0.1529), MO(0.1025), and UK(0.1210).

For this problem, let the augmented matrix for the associated SAAP be denoted by Ω_M. The condition of this matrix is

$$\kappa(\Omega_M) = \|\Omega_M\| \left\|\Omega_M^{-1}\right\| = 2.500 \times 53.820 = 134.5, \tag{37}$$

and clearly, it is ill-conditioned.

The condition of the augmented covariance matrix associated with finding the minimum variance portfolio is

$$\kappa(\Omega_V) = \|\Omega_V\| \left\|\Omega_V^{-1}\right\| = 2.479 \times 63.636 = 157.8 \tag{38}$$

and this matrix is also poorly conditioned. In this regard, DeMiguel et al. [19] have remarked

> "But even the performance of the minimum-variance portfolio depends crucially on the quality of the estimated covariances, and although the estimation error associated with the sample covariances is smaller than that for sample mean returns, it can still be substantial." (p. 799)

Of course, this is really not surprising. If we go back to the geometrical interpretation, the contours of the variance of the portfolio are uncertain and near-linear and the portfolio weight-sum constraint is fixed and linear. Again the two slopes are just about the same over a large neighbourhood; so any small change in the variance contour can lead to a large change in the resulting weights.

Ledoit and Wolf [11] offer significant evidence based on Monte Carlo simulation that the condition of a covariance matrix will fall as its dimension increases. We have mild evidence here that this is the case. But obviously more analysis is required to support this conclusion.

5. Conclusions

Despite the advances in portfolio optimization made since Markowitz, particularly in the last fifteen years, we argue that solution of the Strategic Asset Allocation Problem (a relatively small number of asset classes) is quite sensitive to the input and this sensitivity can be explained by the condition of the augmented covariance matrix. What is more important, there does not appear to be a direct way to get around this ill-conditioning. It remains an open question whether larger portfolio problems suffer from the same problem. The Stein-type estimator offered by Ledoit and Wolf [11] appears to be the best bet to solve the problem. If this larger problem can be solved, then one would simply aggregate weights over asset classes to get the solution of a corresponding Strategic Asset Allocation Problem.

What this analysis really brings into question is the practise of an investment advisor asking a client to answer a series of questions to gauge the client's risk tolerance. Presumably, the answers to these questions will lead to the advisor putting the client into an asset portfolio (i.e. the right combination of money market, bonds, and stock classes) consistent with the client's risk tolerance. Our analysis suggests that there is no reliable science behind this practise.

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